PHYSICAL REVIEW E VOLUME 50, NUMBER 1 JULY 1994

Class of nonsingular exact solutions for Laplacian pattern formation

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(Received 1 June 1993)

We present a class of exact solutions for the so-called Laplacian growth equation describing the zero-surface-tension limit of a variety of two-dimensional pattern formation problems. These solutions are free of finite-time singularities (cusps) for quite general initial conditions. They reproduce various features of viscous fingering observed in experiments and numerical simulations with surface tension, such as existence of stagnation points, screening, tip splitting, and coarsening. In certain cases the asymptotic interface consists of N separated moving Saffman-Taylor fingers.

PACS number(s): 47.15.Hg, 47.20.Hw, 68.10.−m, 68.70.+w

The problem of pattern formation is one of the most rapidly developing branches of nonlinear science today (see, e.g., Ref. [1]). Of special interest is the study of the front dynamics between two phases (interface) that arises in a variety of nonequilibrium physical systems. If, as it usually happens, the motion of the interface is slow in comparison with the processes that take place in the bulk of both phases (such as heat transfer, diffusion, etc.), the scalar field governing the evolution of the interface is a harmonic function. It is natural then, to call the whole process *Laplacian growth.* Depending on the system, this harmonic scalar field is a temperature (in the freezing of a liquid or Stefan problem), a concentration (in solidification from a supersaturated solution), an electrostatic potential (in electrodeposition), a pressure (in flows through porous media), a probability (in diffusion-limited aggregation), etc.

We present, in this paper, a class of solutions of the two-dimensional (2D) Laplacian growth problem in the limit of zero-surface tension. These solutions are quite general, no symmetries of the moving interface are assumed. Most remarkably, they do not develop finite-time singularities but, contrary to the common belief, remain smooth for all times. Furthermore, they are able to reproduce different behaviors observed in experiments, such as tip splitting, screening, and coarsening. Thus, they may describe real fingering instabilities when surface tension is very small, suggesting that surface tension might be treated in this case as a regular perturbation. In certain cases, they give rise, asymptotically in time, to N separated fingers, each of which (for enough separation) describes the Saffman-Taylor finger [2] in channel geometry, and whose evolution closely resembles the N-soliton formation in nonlinear integrable partial differential equations (PDE's).

In the absence of surface tension, whose effect is to stabilize the short-wavelength perturbations of the interface, the problem of 2D Laplacian growth is described as follows:

$$(\partial_x^2+\partial_y^2)u=0\ , \tag{1}$$

$$u|_{\Gamma(t)}=0,\quad \partial_n u|_{\Sigma}=1\ , \tag{2}$$

$$v_n=\partial_n u|_{\Gamma(t)}\ . \tag{3}$$

Here $u(x,y;t)$ is the scalar field mentioned above, $\Gamma(t)$ is the moving interface, Σ is a fixed external boundary, ∂_n is the component of the gradient normal to the boundary (i.e., the normal derivative), and v_n is the normal component of the velocity of the front.

We consider first an infinitely long interface. We introduce then a time-dependent conformal map f from the lower half of a "mathematical" plane, $\zeta\equiv\xi+i\eta$, to the domain of the physical plane, $z\equiv x+iy$, where the Laplace equation (1) is defined, $\zeta\xrightarrow{f}z$. We also require that $f(t,\zeta)\approx\zeta$ for $\zeta\to\xi-i\infty$. Thus, the function $z=f(t,\xi)$ describes the moving interface. Using this conformal map and taking into account the boundary conditions of the problem, we find

$$v_n=\frac{\mathrm{Im}(\bar f_t f_\xi)}{|f_\xi|},\quad \partial_n u_{\Gamma(t)}=\partial_l\psi=\frac{\partial\xi}{|\partial f|}=\frac{1}{|f_\xi|}\ , \tag{4}$$

where the overbar means complex conjugate, the subscripts t and ξ indicate partial derivatives with respect to t and ξ, respectively, ∂_l is the component of the gradient tangent to the interface, ψ is a harmonic function of x and y, conjugate to u, that satisfies $\psi=\xi$ due to the boundary conditions (2) and (3). Equating the right hand sides (rhs) in Eq. (4), in accordance with Eq. (3), we finally obtain

$$\mathrm{Im}(\bar f_t f_\xi)=1,\quad f_\xi|_{\xi-i\infty}=1\ . \tag{5}$$

As in Ref. [3], we will refer to Eq. (5) as the *Laplacian growth equation* (LGE), because the scalar field determining the growth obeys the Laplace Eq. (1). The LGE was first derived, to our knowledge, in 1944 independently by Galin and Polubarinova-Kochina [4]. This equation has time-dependent solutions, unexpected for nonlinear PDE's, such as a set of solutions in the class of polynomials [5] and other exact solutions, though for quite restricted initial shapes [7]. Also the Saffman-Taylor finger [2] is a particular traveling wave solution of this equation. All these properties (except the latter one) are nontrivial and nonperturbative due to the nonlinear nature of the LGE.

Unfortunately, despite these remarkable properties, practically all known solutions of the LGE show finite-time singularities via the formation of cusps [5,6]. Therefore, all these nonperturbative results are helpless to shed light on the physics and geometry of the system in the long-term limit. (Although a few exact results have been presented that have no finite-time cusps [7], they correspond to cases with very restricted symmetries or initial conditions of the moving interface.) As a result it has been generally assumed that these finite-time singularities are an essential feature of LGE solution [8] and that, in this sense, the physics represented by the LGE is incomplete. Thus, the natural attitude was to include surface tension in the theory to stabilize the moving interface and get rid of the finite-time singularities [8]. On the other hand, surface tension was assumed to be unavoidable in order to get certain types of evolution observed in experiments of viscous fingering.

The main result of this paper is to show that the LGE admits quite a broad class of exact time-dependent solutions which *remain smooth for an infinite time* and *reproduce observed phenomena usually attributed to surface tension effects.* Therefore, contrary to the widespread view, zero-surface-tension solutions are able to reproduce nonzero surface-tension evolutions for all times. However, for the selection among our set of solutions, the inclusion of surface tension is indeed unavoidable.

To introduce these solutions, we start from the statement that any function $f(t,\xi)$ whose derivative, f_ζ, has an arbitrary distribution of moving poles, $\zeta_k(t) \equiv \xi_k + i\eta_k$, and roots, $Z_k(t)$, in the upper-half plane, $\mathrm{Im}\zeta > 0$, and no other singularities, such as

$$f_\zeta = \prod_{k=1}^{N+1} \frac{\zeta - Z_k(t)}{\zeta - \zeta_k(t)} , \tag{6}$$

is a solution of the LGE. By substitution of Eq. (4) into Eq. (5) we find

$$f = -it + \zeta - i \sum_{k=1}^{N+1} \alpha_k \ln(\zeta - \zeta_k), \quad \mathrm{Im}\xi \le 0 , \tag{7}$$

where the complex constants of motion α_k and β_k

$$\beta_k \equiv f(\bar{\zeta}_k) = -it + \bar{\zeta}_k - i \sum_{l=1}^{N+1} \alpha_l \ln(\bar{\zeta}_k - \zeta_l) ,$$
$$1 \le k \le N+1 , \tag{8}$$

govern the dynamics of the poles ζ_k, and thus, of the interface. This solution, for $N=1$ corresponds to the development of an isolated finger, similar to the one found by Saffman for channel geometry [9].

The break of analyticity of the interface (a cusp) occurs when at least one of the moving poles, $\zeta_k(t)$, or zeros, $Z_k(t)$, of f_ζ crosses the real axis, $\eta = 0$, of the mathematical plane, ζ. If all ζ_k's and Z_k's remain on the upper-half plane during the whole evolution, then the moving interface remains smooth (analytic) for an infinite time. To obtain sufficient conditions under which this is true for f given by Eqs. (7) and (8), we note the following.

(i) In order for the solution to exist as $t \to \infty$ and satisfy $\eta_k > 0$ for all finite times, all α_k's must have positive real part. This is the only way that the divergent term $-it$ in the r.h.s. of Eq. (8) can be compensated. The term that compensates it is $-i\mathrm{Re}(\alpha_k)\ln(\bar{\zeta}_k - \zeta_k) = -i\mathrm{Re}(\alpha_k)\ln(-2i\eta_k)$ and implies that $\eta_k \to 0$ as $t \to \infty$.

(ii) An isolated singularity, ζ_k, can never reach the real axis at a finite time. If it did, the term $-i\alpha_k \ln(\bar{\zeta}_k - \zeta_k)$ in Eq. (8) would diverge and could not be compensated by any other. On the other hand, if all $\mathrm{Re}\alpha_k$'s are positive, then groups of $M \le N+1$ singularities could not reach the real axis simultaneously at a finite time for exactly the same reason. So, if $\mathrm{Re}\alpha_k > 0$ and $\eta_k(t=0) > 0$ for $1 \le k \le N+1$, then $\eta_k(t) > 0$ for all finite times.

(iii) We assume first that all α_k's are real and positive. After a little algebra, we write the real part of f_ζ as

$$\mathrm{Re} f_\zeta = 1 + \sum_{k=1}^{N+1} \frac{\alpha_k \eta_k}{|\zeta - \zeta_k|^2} - \eta \sum_{k=1}^{N+1} \frac{\alpha_k}{|\zeta - \zeta_k|^2} . \tag{9}$$

Since all α_k's and all η_k $(t=0)$'s are positive, then by the result in (ii), also $\eta_k > 0$ for all finite times. Thus, we see from Eq. (9), that $\mathrm{Re} f_\zeta$ equals zero only if η is strictly positive. This means that the zeros, Z_k, of f_ζ lie always on the upper half of the mathematical plane and never cross the real axis.

Consequently, *if at time $t=0$, all α_k's are real and positive and all η_k's are positive, then the interface represented by Eq. (7) remains smooth throughout its evolution.*

In spite of the lack of a rigorous proof, we believe that also in the general case of complex α_k's with positive real parts, there are no finite-time cusps for a broad range of initial conditions. In fact, we have performed numerous computational experiments in this case and did not encounter cusp formation in any of them. Furthermore, we monitored the movement of the roots and poles of f_ζ in the mathematical plane and they never crossed the real axis. In the long-term limit, we found that roots and poles separate on pairs such that $\mathrm{Im}Z_k$ *always* exceeds the imaginary part of the corresponding pole, ζ_k [which, as proved in (ii), can never reach the real axis if all $\mathrm{Re}\alpha_k$'s are positive]. Thus, we conjecture that solution (7) is free of finite time cusps for all choices of $\zeta_k(0)$ [$\mathrm{Im}\zeta_k(0) > 0$] and α_k ($\mathrm{Re}\alpha_k > 0$), except, possibly, for a set of measure zero [10].

Equation (7) is not the only solution of the LGE that is characterized by the motion of simple poles. For example, if in Eq. (7) we replace $\ln(\zeta - \zeta_k)$ by $\ln(e^{i\zeta} - e^{i\zeta_k})$, and introduce a parameter λ, we find a 2π-periodic solution, relevant for channel geometry, of the form

$$f = -i\frac{t}{\lambda} + \lambda\zeta - i \sum_{k=1}^{N} \alpha_k \ln(e^{i\zeta} - e^{i\zeta_k}) ,$$
$$-\pi \le \mathrm{Re}\zeta \le \pi, \quad \mathrm{Im}\xi \le 0 , \tag{10}$$

where $\lambda \equiv 1 - \Sigma_k \alpha_k$ is the fraction of width of channel occupied by the fingers. This solution has the same properties as Eq. (7). In particular, there exist N constants of motion defined by $\beta_k \equiv f(\bar{\zeta}_k)$ and cusps are absent if all α_k's are real and positive and all η_k's are positive, as it follows from the equation

$$f_{\zeta}=\lambda+\sum_{k=1}^{N}\frac{\alpha_k}{|1-e^{i(\zeta_k-\zeta)}|^2}(1-e^{-\eta_k}e^{i(\zeta-\xi_k)})\neq 0 \qquad \text{if Im}\,\zeta\leq 0\,. \qquad (11)$$

A solution of the LGE similar to Eq. (10), but with $e^{-i\zeta}$ instead of $e^{i\zeta}$, was proposed in [8], where cusps were found via numerical simulations. Taking advantage of the corresponding constants of motion, β_k, we can easily show the necessity of these cusps if $\text{Re}\alpha_k>0$ [10] (as in Ref. [8], where $\alpha_k=1$ for all k).

The constants α_k have a clear geometrical meaning in the physical plane for both solutions (7) and (10): $|\pi\alpha_k|$ and $\arg(\alpha_k)$ are related to the width and the slope of the gap between adjacent moving fingers, in the case of enough separation [10]. We show this property in Fig. 1, where we have plotted the interface $Y\equiv\text{Im}f(t,x)$ vs $X\equiv\text{Re}f(t,x)$, for the solution (7) with two singularities ($N=1$), at a particular time. The real parts of both α's are positive, but while α_1 is purely real, α_2 has a nonvanishing imaginary part. We have drawn on top of each gap a dashed line of length $|\pi\alpha_k|$ and slope $\text{Im}\alpha_k/\text{Re}\alpha_k$ that highlights the meaning of α_k. We conclude then that, if all α_k's are real and positive, the interface is a single-valued function $Y(X)$. Among the experiments on two-dimensional viscous fingering in a channel, one can find different degrees of bending and ramification of the moving fingers [11]. It follows from these experiments that non-single-valued interfaces generally appear. Therefore, complex α's are necessary to describe them using Eq. (7) or Eq. (10). As we show later, we can indeed reproduce these observations by means of solutions (7) and (10) with complex α's.

The constants β_k also have a clear geometrical meaning in the physical plane: the points $(\text{Re}\beta_k, \text{Im}\beta_k+\text{Re}\alpha_k\ln 2)$ are the coordinates of the tips of the gaps between fingers. We show this property in Fig. 2, where we have plotted the interface obtained from Eq. (7) at different times for $N=6$ and real and positive α_k's. For the sake of generality we have deliberately taken the initial condition without any particular symmetry. We have indicated in this figure the location of the tips with asterisks. As one can see, they are "stagnation points" of the interface. These kind of stagnation points have been observed in numerous experiments [11] and numerical simulations [12]. It is interesting to note that we can also identify stagnation points occurring in numerical [13] and physical [14] experiments of diffusion-limited aggregation (DLA) (a feature usually explained as a "screening" effect). This is in accordance with the view (which

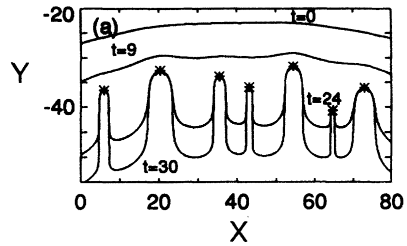

FIG. 2. We plot the interfaces, Y as a function of X, at times $t=0,9,24$, and 30, for f given by Eq. (7) with $\alpha_1=0.80$, $\beta_1=6.00-i37.11$, $\alpha_2=2.00$ $\beta_2=20.44-i33.98$, $\alpha_3=1.00$, $\beta_3=35.61-i34.49$, $\alpha_4=0.50$, $\beta_4=43.15-i36.31$, $\alpha_5=1.50$, $\beta_5=54.58-i32.78$, $\alpha_6=0.35$, $\beta_6=64.65-i40.88$, $\alpha_7=1.80$, and $\beta_7=72.90-i37.40$. In this case the singularities move towards different points on the real axis while the interface develops $N=6$ separated fingers with stagnation points, indicated with asterisks, in between.

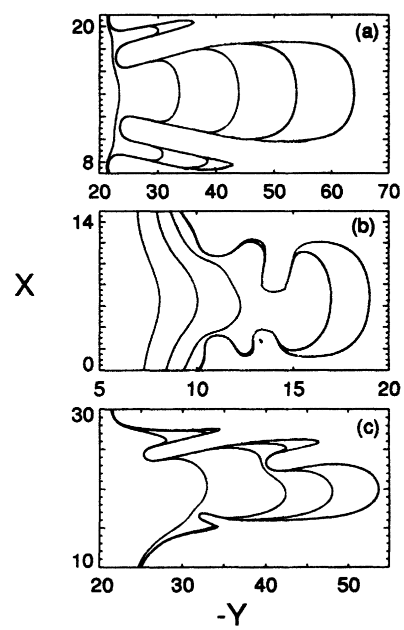

FIG. 3. We plot the interfaces, X as a function of $-Y$, obtained with Eq. (7), for three simulations that reproduce different phenomena observed in experiments: screening and coarsening (a); side branching (b); screening, coarsening and tip splitting (c). The parameters are $\alpha_k=0.7+i0.1$, $0.8+i0.08$, $0.8-i0.1$, $0.7-i0.08$, $0.7+i0.1$, $0.8+i0.08$, $0.8-i0.1$, $0.7-i0.08$; $\zeta_k(0)=-6.4+i7$, $-3.924778+i6$, $i6$, $2.5+i7.5$, $7+i7$, $9.5+i6$, $13.4+i6$, $16.4+i7.5$ (a); $\alpha_k=1+i0.9$, $0.07-i0.5$, $0.06-i0.4$, $0.05+i0.45$, $0.07+i0.7$, $1-i1.15$; $\zeta_k(0)=-5+i5$, $i6$, $0.5+i8$, $4+i8.5$, $7+i6$, $12+i5$ (b); $\alpha_k=3+i0.19$, $0.7-i0.1$, $0.3-i0.11$, $0.8+i0.15$, $0.6+i0.14$, $0.7+i0.13$, $2.5+i0.1$; $\zeta_k(0)=-2+i5$, $-0.3+i6$, $3+i9$, $7.5+i15$, $8.2+i5$, $11.2+i3$, $13+i3$ (c).

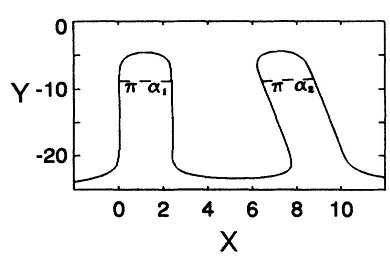

FIG. 1. We plot the interface, $Y\equiv\text{Im}f(t,\xi)$ as a function of $X\equiv\text{Re}f(t,\xi)$ for f defined in Eq. (7) with $\alpha_1=0.8$; $\alpha_2=0.8+i0.1$. The dashed lines that we have drawn on the gaps between fingers have length $|\pi\alpha_k|$ and slope $\text{Im}(\alpha_k)/\text{Re}(\alpha_k)$.

we believe) that DLA fractal growth is described by the LGE in the continuous limit.

Solutions (7) and (10) are also able to reproduce other features observed in experiments, as we show in Fig. 3 where we plotted the interface given by Eq. (7) in different cases. Figure 3(a) exhibits coarsening and screening (a phenomenon analyzed in Ref. [16]), Fig. 3(b), side branching and Fig. 3(c), a combination of tip splitting, coarsening, and screening. We chose the parameters in Fig. 3(c) so as to imitate the evolution displayed in Fig. 5(a) of Ref. [17], which corresponds to an experiment with fairly small surface tension. We can observe that initially two fingers are formed, one of which gets initially broader, but then starts to split, while the other stops growing. Usually these phenomena have been attributed to a competition between surface-tension and the fingering instability [17]. However, these simulations show that surface tension is not necessary to find this type of evolution and hopefully might be treated as a regular perturbation [18]. The main differences between Figs. 2 and 3 are produced by the choice of complex α's and the merging of singularities in the latter (for more details, see Ref. [10]).

Following an argument similar to that of Howison in Ref. [7], it is possible to prove that any smooth initial condition of the LGE can be approximated, to any degree of accuracy, by Eqs. (7) or (10) with an appropriate, but not unique, choice of N, α_k, and $\zeta_k(0)$ [15]. However, initial conditions that generate cusps are dense in this same sense [7]. This reflects a highly unstable situation, in which nearby initial conditions can lead to very different evolutions. Thus, these zero-surface-tension solutions lack predictability and only the inclusion of surface tension would select a unique evolution.

The evolution of the interface in Fig. 2 resembles the N-soliton solution of classical exactly solvable PDE's, where in the long-time asymptotics one can also have N separated solitons, each of which described by the single-soliton solution of the corresponding PDE. An evident difference is that fingers, unlike "classical" solitons, always have a nonzero velocity component normal to the interface. The connection between the N-soliton and the N-finger solutions is deeper than a superficial resemblance. In fact, we believe that the LGE might have an underlying Hamiltonian structure with separation into action-angle variables. We are presently working in this direction.

We thank M. Ancona and R. Camassa for their comments. This work is supported by the U.S. Department of Energy at LANL.

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